

SU3 isoscalar factors

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Abstract

A summary of the properties of the Wigner Clebsch-Gordan coefficients and isoscalar factors for the group SU3 in the $SU2 \otimes U1$ decomposition is presented. The outer degeneracy problem is discussed in detail with a proof of a conjecture (Braunschweig's) which has been the basis of previous work on the SU3 coupling coefficients. Recursion relations obeyed by the SU3 isoscalar factors are produced, along with an algorithm which allows numerical determination of the factors from the recursion relations. The algorithm produces isoscalar factors which share all the symmetry properties under permutation of states and conjugation which are familiar from the SU2 case. The full set of symmetry properties for the SU3 Wigner-Clebsch-Gordan coefficients and isoscalar factors are displayed.

I. Introduction

The group SU3 continues to be useful in modeling symmetries observed in particle and nuclear physics. In the late 1950's it found application in classification of “elementary” hadrons, and in the description of rotational states of non-spherical nuclei. Its utility persists as the color symmetry of quantum chromodynamics, various models for collective nuclear motion, and elsewhere.

The Wigner-Clebsch-Gordan coefficients (WCG) are of particular interest. These can be defined as the expansion coefficients of a composite state of good SU3 quantum numbers in terms of direct products of two individual SU3 classified states, paralleling Wigner's original use of SU2 in the treatment of quantum angular momentum. The WCG can also be developed as the matrix elements of a set of tensor operators which have distinctive properties under the transformations of SU3. These two viewpoints on the WCG are formally identical, and their algebraic connection is expressed by the Wigner-Eckart Theorem. The SU3 case presents a complication that is absent in the SU2 recoupling problem – that of the *outer degeneracy*. The complete determination of WCG's in SU3 requires a criterion outside the SU3 group to completely classify composite states, and thus to fully define numerical values for the WCG. In *this* process, the two perspectives on the WCG mentioned above suggest quite different mechanisms.

Biedenharn and coworkers [1, 2, 3], adopting the operator point of view, have developed a set of canonical SU3-labeled unit tensor operators, whose matrix elements become the “canonical WCG's.” The canonical operators acquire SU3 labels by virtue of their behavior under the transformations of the group. As well, each produces a unique set of shifts – i.e, its action when operating on a state from a particular irreducible representations (irrep) produces states from a unique second irrep; and in cases of non-trivial outer degeneracy there is a distinct operator for each degeneracy index. The uniqueness of the operators, and thus their designation as canonical, comes from their null space properties. The *characteristic null space* of an operator is the union of all irreps which identically yield zero under the action of the operator. In the case of a tensor operator of degeneracy one, the null space is uniquely determined by the group properties of the operator and the state operated upon: for higher degeneracy the operators for distinct degeneracy labels are chosen to have a null space each larger than the previous and completely containing it.

Adopting the alternative viewpoint – the WCG's as coupling coefficients which give the amplitude for the joining of two SU3 states to a composite state of good SU3 quantum numbers – coefficients which exhibit symmetry under interchange of the two states being coupled are suggested (a symmetry missing from the canonical coefficients [4].) WCG's for those couplings which have degeneracy 1 possess this symmetry, and indeed all the symmetries under permutation of irreps of the familiar SU2 Clebsch-Gordan coefficients. It has been proven that such permutation symmetric WCG's for the SU3 case with degeneracy > 1 exist [5, 6, 7] and examples of such SU3 WCG's have been developed [8, 9].

In the sections which follow, new results pertaining to the SU3 WCG's are presented which simplify evaluation of the WCG's, and which are independent of the particular scheme adopted

for outer degeneracy resolution. In particular, a collection of recursion relations are defined for the isoscalar factors. An algorithm is presented which utilizes these recursion relations to generate a set of WCG's demonstrating all the Racah symmetries familiar from SU2. This algorithm has been used in a successful C language implementation.

II. Definitions and Notation

A linear vector space which carries an irrep of SU3 is fully specified by two integers (p, q) henceforth referred to as the *irrep labels*. The dimension of the space is

$$d = (p+1)(q+1)(p+q+2)/2. \quad (1)$$

A complete set of d orthogonal vectors within the irrep can be labeled by three further integers (k, l, m) , the *subspace labels*, which satisfy the *betweenness conditions*

$$p+q \geq k \geq q \geq l \geq 0 ; k \geq m \geq l. \quad (2)$$

A fully specified member of the orthonormal spanning set for the irrep is denoted by the ket

$$|p, q; k, l, m\rangle.$$

The WCG are the coefficients (C) of the expansion of a composite SU3 state ket in terms of products of SU3 kets

$$|\mathcal{P}; \kappa\rangle = \sum C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} | \mathcal{P}_1; \kappa_1\rangle | \mathcal{P}_2; \kappa_2\rangle. \quad (3)$$

where \mathcal{P} is shorthand for the pair (p, q) and κ for the set (k, l, m) ; the sum extends over subspace labels of κ_1 and κ_2 ; and $n = 0, 1, \dots$ labels the outer degeneracy. The Wigner-Eckart Theorem relates the WCG's to matrix elements of operators $T_{p,q;k,l,m}^n$ which transform like tensors under the operations of SU3

$$\begin{aligned} \langle p, q; k, l, m | T_{p_1, q_1; k_1, l_1, m_1}^n | p_2, q_2; k_2, l_2, m_2 \rangle = \\ C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} < p, q || T_{p_1, q_1}^n || p_2, q_2 >, \end{aligned} \quad (4)$$

where

$$< p, q || T_{p_1, q_1} || p_2, q_2 >,$$

the *reduced matrix element*, is a complex number which depends only upon the three sets of irrep labels. (The unit tensor operators of the Biedenharn scheme are so named since each has a reduced matrix element of one.)

The ket labeling scheme described above represents the decomposition $SU3 \supset SU2 \otimes U1$. The labels (k, l, m) are the quantum numbers of the $SU2$ subgroup, and are related to the isospin (I) and its z component (I_z) by

$$I = \frac{k-l}{2} \quad (5)$$

$$I_z = m - \frac{k+l}{2}; \quad (6)$$

and the $U1$ subgroup with the hypercharge (Y) given by

$$Y = k + l - \frac{2}{3}(p + 2q). \quad (7)$$

The WCG of equation (3) will vanish unless the subspace labels obey the relations

$$I_1 + I_2 \geq I, \quad (8)$$

$$|I_1 - I_2| \leq I, \quad (9)$$

$$I_{1z} + I_{2z} = I_z, \quad (10)$$

$$Y_1 + Y_2 = Y. \quad (11)$$

This decomposition allows factoring of an $SU2$ Clebsch-Gordan coefficient from the $SU3$ WCG as follows:

$$C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} = C \begin{smallmatrix} I_1 \\ I_{1z} \end{smallmatrix} \begin{smallmatrix} I_2 \\ I_{2z} \end{smallmatrix} \begin{smallmatrix} I \\ I_z \end{smallmatrix} F^n(p, q, k, l : p_1, q_1, k_1, l_1; p_2, q_2, k_2, l_2) \quad (12)$$

where the factor F , which is independent of the m subspace labels, is called the *isoscalar factor* (ISF). In subsequent usage, when their values are obvious from the context, the p and q values will be suppressed in the notation for the ISF.

A particular set of subspace labels, $k = m = p + q, l = 0$, will play an important role in the present consideration. This set, referred to as the *state of highest weight* for a particular irrep, will be referred to by the replacement $(k = p + q, l = 0, m = p + q) \rightarrow SHW$ and likewise in the isoscalar factor by $(p, q, k = p + q, l = 0) \rightarrow (p, q, SHW)$.

III. Outer degeneracy

The Clebsch-Gordan series for $SU3$

$$(p_1, q_1) \otimes (p_2, q_2) = \sum_i \eta_i (p'_i, q'_i) \quad (13)$$

indicates the number of distinct times (η_i) the irrep (p'_i, q'_i) appears in the outer product of irreps (p_1, q_1) and (p_2, q_2) . The circumstance of $\eta_i > 1$ is a feature of $SU3$ referred to as *outer degeneracy*, and the coefficients η_i will be referred to herein as the *degeneracy* of the coupling $(p_1, q_1) \otimes (p_2, q_2) \rightarrow (p'_i, q'_i)$.

The value of the degeneracy is a function of the six irrep labels and can be deduced from the betweenness conditions of equation (2) for each irrep, and the requirements of the $SU2$ and $U1$ subgroups, given in equations (8), (9) and (10). These latter requirements follow from the corresponding Clebsch-Gordan series for $SU2$ (triangularity of three Euclidian vectors in two dimensions) and for $U1$ (scalar addition.) Various ways of evaluating the degeneracy appear in the literature [10, 11]. An equivalent expression for the degeneracy consistent with present notation is

$$\begin{aligned} \eta &= \max(\eta' + 1 - \max(\gamma, \sigma), 0) \\ \eta' &= \min(p_1 + \sigma, p_2 + \sigma, q + \sigma, q_1 + \gamma, q_2 + \gamma, p + \gamma, 2(\sigma + \gamma), \\ &\quad p_1 + q_1 - \gamma - \sigma, p_2 + q_2 - \gamma - \sigma) \end{aligned} \quad (14)$$

where $\gamma \equiv (p_1 + p_2 - p)/3$, and $\sigma \equiv (q_1 + q_2 - q)/3$.

This expression can be used to prove Braunschweig's conjecture which has been used by several authors [12, 13, 14] in work related to determination of WCG's for SU3. The conjecture suggests that the number of non-vanishing values of the WCG

$$C \begin{matrix} [\mathcal{P}_1] \\ [SHW] \end{matrix} \begin{matrix} [\mathcal{P}_2] \\ [\kappa_2] \end{matrix} \begin{matrix} [\mathcal{P}] \\ [SHW] \end{matrix} \quad (14)$$

is no less than the degeneracy of the irrep coupling. The value for the subspace label m_2 is fixed by equation (9) and l_2 is dependent upon k_2 through equation (10)

$$k_2 + l_2 = p_2 + q_2 - \gamma + \sigma, \quad (15)$$

so counting the number of non-vanishing WCG's of this type can be accomplished by determining the range of k_2 values. Upper and lower limits on k_2 come from the triangularity expressions of equation (8) combined with equation (15), producing

$$\begin{aligned} k_2 &\geq p_2 + q_2 - 2\gamma - \sigma \\ k_2 &\geq \gamma + 2\sigma \\ k_2 &\leq p + q + \gamma + 2\sigma. \end{aligned} \quad (16)$$

The betweenness relations for state 2 give further limits on k_2 ($p_2 + q_2 \geq k_2 \geq q_2$) and on l_2 ($q_2 \geq l_2 \geq 0$), which when combined with equation (15) produce

$$\begin{aligned} k_2 &\geq \sigma - \gamma + p_2, \\ k_2 &\leq \sigma - \gamma + p_2 + q_2. \end{aligned} \quad (17)$$

The limits on k_2 are thus

$$\max(p_2 + q_2 - 2\gamma - \sigma, \gamma + 2\sigma, q_2, p_2 + \sigma - \gamma) \leq k_2 \leq \min(p_2 + q_2 + \sigma - \gamma, p_2 + q_2, p + q + \gamma + 2\sigma) \quad (18)$$

which implies that the total number of k_2 values producing non-vanishing WCG's in this case is given by

$$1 + \min \left(\gamma + 2\sigma, \sigma + 2\gamma, p_2 + q_2 - \sigma - 2\gamma, p_2 + q_2 - \gamma - 2\sigma, q_2, \right. \\ \left. q_2 + \gamma - \sigma, p_1 + q_1, p + q, p + q - q_2 + \gamma + 2\sigma, p + q - p_2 + 2\gamma + \sigma \right). \quad (19)$$

A term-by-term comparison of this expression with that for the degeneracy (equation (13)) reveals that the number of k_2 values producing non-vanishing WCG's of the form of equation (14) is greater than or equal to the degeneracy. This inequality is sufficient for the uses of the previously cited references, and has been important in the development of the algorithm presented later in this paper.

III. Recursion relations

An efficient scheme for evaluation of the SU3 ISF *independent of the method chosen for resolution of outer degeneracy* involves use of recursion relations for these quantities, which can

be derived using the group generators [15]. Explicit expressions for the generators depend upon a choice of signs of the matrix elements of the generators between elements of the fundamental three-dimensional representation: those given below follow the phase convention of de Swart [16]. The actions of these generators on the orthonormal kets previously defined are

$$\hat{T}_+ |p, q; k, l, m\rangle = \sqrt{(k-m)(m-l+1)} |p, q; k, l, m+1\rangle \quad (20)$$

$$\hat{T}_- |p, q; k, l, m\rangle = \sqrt{(k-m+1)(m-l)} |p, q; k, l, m-1\rangle \quad (21)$$

$$\begin{aligned} \hat{V}_+ |p, q; k, l, m\rangle &= \sqrt{\frac{(k+2)(m-l+1)(k-q+1)(p+q-k)}{(k-l+1)(k-l+2)}} |p, q; k+1, l, m+1\rangle \\ &+ \sqrt{\frac{(l+1)(k-m)(q-l)(p+q-l+1)}{(k-l)(k-l+1)}} |p, q; k, l+1, m+1\rangle \end{aligned} \quad (22)$$

$$\begin{aligned} \hat{V}_- |p, q; k, l, m\rangle &= \sqrt{\frac{(k+1)(m-l)(k-q)(p+q-k+1)}{(k-l)(k-l+1)}} |p, q; k-1, l, m-1\rangle \\ &+ \sqrt{\frac{l(k-m+1)(q-l+1)(p+q-l+2)}{(k-l+1)(k-l+2)}} |p, q; k, l-1, m-1\rangle \end{aligned} \quad (23)$$

$$\begin{aligned} \hat{U}_+ |p, q; k, l, m\rangle &= \sqrt{\frac{(k+2)(k-m+1)(k-q+1)(p+q-k)}{(k-l+1)(k-l+2)}} |p, q; k+1, l, m\rangle \\ &- \sqrt{\frac{(m-l)(l+1)(q-l)(p+q-l+1)}{(k-l)(k-l+1)}} |p, q; k, l+1, m\rangle \end{aligned} \quad (24)$$

$$\begin{aligned} \hat{U}_- |p, q; k, l, m\rangle &= \sqrt{\frac{(k+1)(k-m)(k-q)(p+q-k+1)}{(k-l)(k-l+1)}} |p, q; k-1, l, m\rangle \\ &- \sqrt{\frac{l(m-l+1)(q-l+1)(p+q-l+2)}{(k-l+1)(k-l+2)}} |p, q; k, l-1, m\rangle. \end{aligned} \quad (25)$$

Three diagonal operators indicate the values of the subspace labels for a ket:

$$\hat{T}_3 |p, q; k, l, m\rangle = I_z |p, q; k, l, m\rangle \quad (26)$$

$$\hat{Y} |p, q; k, l, m\rangle = Y |p, q; k, l, m\rangle \quad (27)$$

$$\begin{aligned} \hat{T}^2 |p, q; k, l, m\rangle &= \frac{1}{2}(\hat{T}_+ \hat{T}_- + \hat{T}_- \hat{T}_+) |p, q; k, l, m\rangle \\ &= I(I+1) |p, q; k, l, m\rangle \end{aligned} \quad (28)$$

making use of the definitions of equations (6) and (7). The operators \hat{T}_+ and \hat{T}_- move up and down in the variable I_z , and thus have no effect on the ISF. The remaining four nondiagonal (ladder) operators form the basis of the derivation of the recursion relations.

Consider a composite state of highest weight

$$|\mathcal{P}; SHW\rangle = \sum C \begin{bmatrix} \mathcal{P}_1 \\ \kappa_1 \end{bmatrix} \begin{bmatrix} \mathcal{P}_2 \\ \kappa_2 \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ SHW \end{bmatrix} |\mathcal{P}_1; \kappa_1\rangle |\mathcal{P}_2; \kappa_2\rangle. \quad (29)$$

The action of \hat{V}_+ on this ket must vanish since each of the two states it produces have $m = p + q + 1$ which violates betweenness. Linearity of the generators (e.g. $\hat{V}_+ = \hat{V}_{1+} + \hat{V}_{2+}$) implies from equation (29) that

$$\begin{aligned}\hat{V}_+ |\mathcal{P}; SHW\rangle &= 0 \\ &= \sum C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} n \\ [SHW] \end{smallmatrix} \left(|\mathcal{P}_2; \kappa_2\rangle \hat{V}_{1+} |\mathcal{P}_1; \kappa_1\rangle + |\mathcal{P}_1; \kappa_1\rangle \hat{V}_{2+} |\mathcal{P}_2; \kappa_2\rangle \right) \end{aligned} \quad (30)$$

Use of the defining equation for \hat{V}_+ (equation (22)) changes this expression into a summed four-term expression which must vanish. The orthogonality of any two SU3 kets with different subspace labels allows this sum to be transformed into a four term recursion relation for the WCG:

$$\begin{aligned}0 &= \sqrt{\frac{(k_1 + 1)(m_1 - l_1)(k_1 - q_1)(p_1 + q_1 - k_1 + 1)}{(k_1 - l_1)(k_1 - l_1 + 1)}} C \begin{smallmatrix} [\mathcal{P}_1] \\ [k_1 - 1, l_1, m_1 - 1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [k_2, l_2, m_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [SHW] \end{smallmatrix} \\ &+ \sqrt{\frac{(k_2 + 1)(k_2 - q_2)(p_2 + q_2 - k_2 + 1)(m_2 - l_2 + 1)}{(k_2 - l_2)(k_2 - l_2 + 1)}} C \begin{smallmatrix} [\mathcal{P}_1] \\ [k_1, l_1, m_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [k_2 - 1, l_2, m_2 - 1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [SHW] \end{smallmatrix} \\ &+ \sqrt{\frac{l_1(q_1 - l_1 + 1)(k_1 - m_1 + 1)(p_1 + q_1 - l_1 + 2)}{(k_1 - l_1 + 1)(k_1 - l_1 + 2)}} C \begin{smallmatrix} [\mathcal{P}_1] \\ [k_1, l_1 - 1, m_1 - 1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [k_2, l_2, m_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [SHW] \end{smallmatrix} \\ &+ \sqrt{\frac{l_2(k_2 - m_2 + 1)(q_2 - l_2 + 1)(p_2 + q_2 - l_2 + 2)}{(k_2 - l_2 + 1)(k_2 - l_2 + 2)}} C \begin{smallmatrix} [\mathcal{P}_1] \\ [k_1, l_1, m_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [k_2, l_2 - 1, m_2 - 1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [SHW] \end{smallmatrix} \quad (31) \end{aligned}$$

This should be valid for any set of projection quantum numbers, thus for $k_1 = m_1$ which allows the replacement of the WCG in this expression by products of ISF and simple SU2 Clebsch-Gordan coefficients whose values can be expressed analytically [17]. The result of this replacement is a four term recursion relation among the ISF's for coupling to a state of highest weight:

$$\begin{aligned}0 &= a_1 F^n(SHW : k_1 - 1, l_1; k_2, l_2) + a_2 F^n(SHW : k_1, l_1; k_2 - 1, l_2), \\ &- a_3 F^n(SHW : k_1, l_1 - 1; k_2, l_2) + a_4 F^n(SHW : k_1, l_1; k_2, l_2 - 1), \end{aligned}$$

where

$$\begin{aligned}a_1 &= \sqrt{\frac{(k_1 + 1)(k_1 - q_1)(p_1 + q_1 - k_1 + 1)(p + q + 2I_1 + 2I_2 + 3)(p + q + 2I_1 - 2I_2 + 1)}{I_1(2I_1 + 1)}}, \\ a_2 &= \sqrt{\frac{(k_2 + 1)(k_2 - q_2)(p_2 + q_2 - k_2 + 1)(p + q + 2I_1 + 2I_2 + 3)(p + q - 2I_1 + 2I_2 + 1)}{I_2(2I_2 + 1)}}, \\ a_3 &= \sqrt{\frac{l_1(q_1 - l_1 + 1)(p_1 + q_1 - l_1 + 2)(-p - q + 2I_1 + 2I_2 + 1)(p + q - 2I_1 + 2I_2 + 1)}{(2I_1 + 1)(I_1 + 1)}}, \\ a_4 &= \sqrt{\frac{l_2(q_2 - l_2 + 1)(p_2 + q_2 - l_2 + 2)(-p - q + 2I_1 + 2I_2 + 1)(p + q + 2I_1 - 2I_2 + 1)}{(2I_2 + 1)(I_2 + 1)}}. \quad (32) \end{aligned}$$

Similarly,

$$\hat{U}_- |\mathcal{P}; SHW\rangle = 0 \quad (33)$$

since the two kets produced by this operation both violate betweenness. By an analogous set of steps one derives a second, distinct recursion relation:

$$\begin{aligned} 0 = & b_1 F(SHW : k_1 + 1, l_1; k_2, l_2) - b_2 F(SHW : k_1, l_1; k_2 + 1, l_2), \\ & + b_3 F(SHW : k_1, l_1 + 1; k_2, l_2) + b_4 F(SHW : k_1, l_1; k_2, l_2 + 1), \end{aligned}$$

where

$$\begin{aligned} b_1 &= \sqrt{\frac{(k_1 + 2)(k_1 - q_1 + 1)(p_1 + q_1 - k_1)(-p - q + 2I_1 + 2I_2 + 1)(p + q - 2I_1 + 2I_2 + 1)}{(2I_1 + 1)(I_1 + 1)}}, \\ b_2 &= \sqrt{\frac{(k_2 + 2)(k_2 - q_2 + 1)(p_2 + q_2 - k_2)(-p - q + 2I_1 + 2I_2 + 1)(p + q + 2I_1 - 2I_2 + 1)}{(2I_2 + 1)(I_2 + 1)}}, \\ b_3 &= \sqrt{\frac{(l_1 + 1)(q_1 - l_1)(p_1 + q_1 - l_1 + 1)(p + q + 2I_1 + 2I_2 + 3)(p + q + 2I_1 - 2I_2 + 1)}{I_1(2I_1 + 1)}}, \\ b_4 &= \sqrt{\frac{(l_2 + 1)(q_2 - l_2)(p_2 + q_2 - l_2 + 1)(p + q + 2I_1 + 2I_2 + 3)(p + q - 2I_1 + 2I_2 + 1)}{I_2(2I_2 + 1)}}. \quad (34) \end{aligned}$$

One can "step down" from the ISF's for the coupled state of highest weight to ISF's for any other k, l values by use of two relations derived in an analogous fashion from the actions of the operators \hat{V}_- and \hat{U}_+ , respectively:

$$\begin{aligned} F^n(k, l : k_1, l_1; k_2, l_2) = & \alpha(c_1 F^n(k + 1, l - 1 : k_1, l_1; k_2, l_2) + c_2 F^n(k, l - 1 : k_1, l_1 - 1; k_2, l_2) \\ & - c_3 F^n(k, l - 1 : k_1, l_1; k_2 - 1, l_2) + c_4 F^n(k, l - 1 : k_1, l_1; k_2, l_2 - 1)) \quad (35) \end{aligned}$$

where

$$\begin{aligned} \alpha &= \frac{k - l + 2}{\sqrt{2l(q - l + 1)(p + q - l + 2)}} \\ c_1 &= \sqrt{\frac{(k + 2)(k - q + 1)(p + q - k)(I_1 + I_2 - I)(-I_1 + I_2 + I + 1)}{(I + 1)^2(I_1 + I_2 + I + 2)(I_1 - I_2 + I + 1)}} \\ c_2 &= \sqrt{\frac{4l_1(q_1 - l_1 + 1)(p_1 + q_1 - l_1 + 2)(2I_1 + 1)}{(2I_1 + 2)(2I_1 + 2I_2 + 2I + 4)(2I_1 - 2I_2 + 2I + 2)}} \\ c_3 &= \sqrt{\frac{(k_2 + 1)(k_2 - q_2)(p_2 + q_2 - k_2 + 1)(I_1 + I_2 - I)}{I_2(2I_2 + 1)(I_1 - I_2 + I + 1)}} \\ c_4 &= \sqrt{\frac{l_2(q_2 - l_2 + 1)(p_2 + q_2 - l_2 + 2)(-I_1 + I_2 + I + 1)}{(2I_2 + 1)(I_2 + 1)(I_1 + I_2 + I + 2)}}; \end{aligned}$$

and

$$\begin{aligned} F^n(k, 0 : k_1, l_1; k_2, l_2) = & \beta(d_1 F^n(k + 1, 0 : k_1 + 1, l_1; k_2, l_2) \\ & + d_2 F^n(k + 1, 0 : k_1, l_1; k_2 + 1, l_2) + d_3 F^n(k + 1, 0 : k_1, l_1; k_2, l_2 + (\mathbb{B}\mathbb{B})) \end{aligned}$$

where

$$\begin{aligned}
\beta &= \sqrt{\frac{(k+2)}{2(k-q+1)(p+q-k)}} \\
d_1 &= \sqrt{\frac{(k_1+2)(k_1-q_1+1)(p_1+q_1-k_1)(2I_1+1)}{(I_1+1)(I_1+I_2+I+2)(I_1-I_2+I+1)}} \\
d_2 &= \sqrt{\frac{(k_2+2)(k_2-q_2+1)(p_2+q_2-k_2)(-I_1+I_2+I+1)}{(2I_2+1)(I_2+1)(I_1+I_2+I+2)}} \\
d_3 &= \sqrt{\frac{(l_2+1)(q_2-l_2)(p_2+q_2-l_2+1)(I_1+I_2-I)}{I_2(2I_2+1)(I_1-I_2-I+1)}}.
\end{aligned}$$

The second of these two expressions is not the most such general relation which can be derived, but when used in combination with the first, it is sufficient to determine the value of any ISF for the given irrep coupling, once the values of the ISF's for $k, l = SHW$ are known.

IV. Determining the isoscalar factors

To move from the recursion relations to determination of the ISF, a sign convention and a resolution scheme for outer degeneracy must be chosen. For a given coupling,

$$(p_1, q_1) \otimes (p_2, q_2) \rightarrow (p, q)$$

the degeneracy is determined by equation (13). For cases of degeneracy $\eta = 1$, the choice of sign of one of the nonvanishing ISF's for $(k, l) = SHW$ is sufficient to determine all the others. In practice, one such ISF is set equal to 1; equations (32) and (34) are used to generate all others from this one; and all are multiplied by a common factor to enforce the normalization condition

$$\sum_{k_1, l_1, k_2, l_2} (F(SHW : k_1, l_1; k_2, l_2))^2 = 1. \quad (37)$$

Even in such simple cases, the particular ISF to initialize must be chosen as one which allows use of the recursion relations (32) and (34) to determine neighboring values, and a recursive path from the starting point to arbitrary ISF's must be deduced.

When the $(k, l) = SHW$ ISF's are all known, equation (36) can be used to deduce all $k < p + q, l = 0$ ISF's, and from them equation (35) implies all $l > 0$ cases.

The most delicate problem is the determination of an algorithm which uniquely determines all ISF's in cases of outer degeneracy two or higher. Such an algorithm has been developed and implemented in C language codes for evaluation of arbitrary ISF's as floating point values, and as exact precision square roots of a ratio of integers [18]. The ISF's produced by this algorithm share all the symmetries under irrep exchange and conjugation with the familiar SU2 Clebsch-Gordan coefficients.

The logic of the algorithm can be made clearer through a change of variables. Of the four integers, k_1, l_1, k_2, l_2 , used heretofore as parameters of the isoscalar factor for a coupling to a

state of highest weight, only three are independent. The hypercharge conservation relation (10) implies

$$k_1 + l_1 + k_2 + l_2 = \frac{1}{3}(2(p_1 + p_2) + 4(q_1 + q_2) + p - q).$$

Use of the definition

$$s \equiv k_1 - l_1 + k_2 - l_2$$

allows the ISF's to be expressed as $F^n(SHW : s, k_1, l_1)$. The degeneracy index $n = 0, 1, \dots, \eta - 1$ is necessary for couplings with degeneracy $\eta > 1$. The algorithm works as follows:

- make the following assignments for $0 \leq n < \eta$, $0 \leq n' < \eta$
 $F^n(SHW : s_{max} - 2n', k_{1min}, l_{1min}) = \delta_{n,n'}$ where δ is the Kroneker delta.
- using these assignments, the recursion relations (32) and (34) are adequate to determine all ISF's (with $k, l = SHW$) for $s_{max} \geq s \geq s_{max} - 2(\eta - 1)$.
- ISF's (with $k, l = SHW$) for remaining values of s can be determined without further assignments, evaluating each set of values for fixed s before moving to lower s . To move to a lower s value, the recursion relation reduces to only three terms either at $k_1 = k_{1max}$, $l_1 = l_{1min}$; or at $k_1 = k_{1min}$, $l_1 = l_{1max}$. This allows determination of one ISF for the new s value, which then allows the evaluation of all others at this s value using the full four-term recursion relation. This stepdown procedure fails in a small subset of cases, whereupon one must take advantage of permutation symmetry (discussed below) to move to the lower value of s .
- The $F^0(SHW : s, k_1, k_2)$ values, once normalized, are proper isoscalar factors. A linear combination of the F^0 's and the F^1 's, made orthogonal to F^0 and normalized, become proper $F^1(SHW : s, k_1, k_2)$ values. Likewise, using the Gram-Schmidt orthogonalization procedure, each set of F^n with higher n is constructed from those with lower n 's.
- Remaining ISF's for values of $k, l \neq SHW$ can be determined in a straightforward way using the remaining recursion relations, (35) and (36).

In this description, s_{max} is the maximum value of s for which a non-vanishing ISF occurs for a coupling of fixed p, q, p_1, q_1, p_2, q_2 values: $k_{1min}, l_{1min}, k_{1max}, l_{1max}$ are the minimum and maximum values of the k_1 and l_1 variables for a particular value of s .

V. Symmetries of the isoscalar factor

The symmetries of the SU2 Clebsch-Gordan coefficient under permutation of irreps (j_i, m_i) and under conjugation ($j, m \rightarrow j, -m$), known as the Racah symmetries, are well known and frequently utilized to simplify tabulations of coefficients and recoupling calculations. For SU3 couplings of degeneracy one, a complete set of Racah symmetries can be demonstrated. The algorithm described in the previous section extends these symmetries to couplings of arbitrary degeneracy, in contrast to some other degeneracy resolution schemes. Once the symmetries of

the SU3 WCG are known, one can use the symmetry relations for the SU2 Clebsch-Gordan coefficients to deduce symmetry properties of the SU3 isoscalar factors.

Derivation of the symmetry relations for the SU3 WCG is straightforward, albeit tedious. If one applies the V_+ operator (equation (22)) to both sides of the defining expression for the WCG (equation (3)), the result is a linear expression involving six WCG's of various indices and corresponding coefficients which sum to zero. Under each of the transformations

1. $(p_1, q_1; k_1, l_1, m_1) \leftrightarrow (p_2, q_2; k_2, l_2, m_2)$, referred to as $(1 \leftrightarrow 2)$;
2. $(p_1, q_1; k_1, l_1, m_1) \rightarrow (q, p; p+q-l, p+q-k, p+q-m)$ and $(p, q; k, l, m) \rightarrow (q_1, p_1; p_1+q_1-l_1, p_1+q_1-k_1, p_1+q_1-m_1)$, referred to as $(1 \leftrightarrow \tilde{3})$; and
3. $(p, q; k, l, m) \rightarrow (q, p; p+q-l, p+q-k, p+q-m)$ and similarly for the states p_1, q_1 and p_2, q_2 , referred to as *conjugation*,

the coefficients in the six term expression of WCG's transform among one another in pairs, easily exhibiting the fact that the transformed WCG's (within a sign which depends on the k, l, m indices) obey the same six term recursion relations as the original WCG's.

The symmetry transformations can also involve a sign change which depends upon the p and q variables. This is fully dependent upon another sign convention which must be chosen in order to fully specify the WCG's and ISF's. Convenient choices are:

$$C_{m_1=j_1, m_2=-j_2}^{j_1, j_2, j} > 0$$

the familiar Condon and Shortley phase convention [19], and

$$F^n(SHW : SHW_1; k_{2max}, l_{2min}) > 0.$$

with the s value of the F chosen to be positive given as $s_{max} - n$. To derive the p, q dependence of the phase under one of the symmetry transformations, begin with a SU3 WCG which is positive under this convention; apply the transformation; and determine the sign of the transformed WCG relative to that which is positive by convention among the transformed coefficients, using the recursion relations derived earlier. This sign, which will depend upon the six p, q values only, becomes part of the symmetry relation.

The absolute magnitude of the ratio of a WCG to its permuted version results from the normalization condition

$$\sum_{\kappa_1, \kappa_2} (C_{\kappa_1}^{[\mathcal{P}_1]} \kappa_1^n [\mathcal{P}_2]_{\kappa_2} [\mathcal{P}]_{\kappa})^2 = 1. \quad (38)$$

It is straightforward to show that two of the transformations – $(1 \leftrightarrow 2)$ and conjugation – produce no change in normalization and thus require no constant term; and the third – $(1 \leftrightarrow \tilde{3})$ – requires a constant equal to the square root of the ratio of the dimension of the irreps \mathcal{P}_1 and \mathcal{P} .

When one considers couplings which have degeneracy greater than one, each set of distinct WCG's (labeled by the index n) has a unique element chosen as positive by convention. As

a result, there is an additional phase contribution of $(-1)^n$ in each of the transformations considered here.

The resulting symmetry relations are as follows:

$$C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} = (-1)^{\gamma+\sigma+\max(\gamma,\sigma)+n} C \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} \quad (39)$$

for the $(1 \leftrightarrow 2)$ transformation;

$$C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} = (-1)^{m_2+n} \sqrt{\frac{(p+1)(q+1)(p+q+2)}{(p_1+1)(q_1+1)(p_1+q_1+2)}} C \begin{smallmatrix} [\tilde{\mathcal{P}}] \\ [\tilde{\kappa}] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\tilde{\mathcal{P}}_1] \\ [\tilde{\kappa}_1] \end{smallmatrix} \quad (40)$$

for the $(1 \leftrightarrow \tilde{3})$ transformation (where $\tilde{\mathcal{P}}$ represents the ordered pair q, p and $\tilde{\kappa}$ represents $p+q-l, p+q-k, p+q-m$); and

$$C \begin{smallmatrix} [\mathcal{P}_1] \\ [\kappa_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}] \\ [\kappa] \end{smallmatrix} = (-1)^{\gamma+\sigma+\min(\gamma,\sigma)+n} C \begin{smallmatrix} [\tilde{\mathcal{P}}_1] \\ [\tilde{\kappa}_1] \end{smallmatrix} \begin{smallmatrix} [\mathcal{P}_2] \\ [\kappa_2] \end{smallmatrix} \begin{smallmatrix} [\tilde{\mathcal{P}}] \\ [\tilde{\kappa}] \end{smallmatrix} \quad (41)$$

for the conjugation transformation.

No additional information is given by repeating this procedure with the ladder operators \hat{V}_- and \hat{U}_+ since they are related to the two operators already considered by Hermitian conjugation: similar treatment using \hat{T}_+ and \hat{T}_- generate the symmetry relations for the SU2 Clebsch-Gordan coefficients.

Corresponding symmetry relations for the isoscalar factors follow from the above combined with the symmetry relations for the SU2 Clebsch-Gordan coefficients in the Condon and Shortley phase conventions. They are

$$F(\mathcal{P}, \kappa : \mathcal{P}_1, \kappa_1; \mathcal{P}_2, \kappa_2) = (-1)^{\gamma+\sigma+\max(\gamma,\sigma)+I_1+I_2-I} F(\mathcal{P}, \kappa : \mathcal{P}_2, \kappa_2; \mathcal{P}_1, \kappa_1) \quad (42)$$

for the $(1 \leftrightarrow 2)$ transformation (here κ represents the pair (k, l) ;

$$F(\mathcal{P}, \kappa : \mathcal{P}_1, \kappa_1; \mathcal{P}_2, \kappa_2) = (-1)^{l_2} \sqrt{\frac{(p+1)(q+1)(p+q+2)(k_1-l_1+1)}{(p_1+1)(q_1+1)(p_1+q_1+2)(k-l+1)}} F(\tilde{\mathcal{P}}_1, \tilde{\kappa}_1 : \tilde{\mathcal{P}}, \tilde{\kappa}; \mathcal{P}_2, \kappa_2) \quad (43)$$

for the $(1 \leftrightarrow \tilde{3})$ transformation; and

$$F(\mathcal{P}, \kappa : \mathcal{P}_1, \kappa_1; \mathcal{P}_2, \kappa_2) = (-1)^{\gamma+\sigma+\max(\gamma,\sigma)+I_1+I_2-I} F(\tilde{\mathcal{P}}, \tilde{\kappa} : \tilde{\mathcal{P}}_1, \tilde{\kappa}_1; \tilde{\mathcal{P}}_2, \tilde{\kappa}_2) \quad (44)$$

for the conjugation transformation.

The choice of a resolution procedure which yields symmetries such as these produces considerable simplifications in calculations of physical states, reduces the complexity of definitions of $6-j$ and $9-j$ type recoupling coefficients, and significantly shortens databases and printed tables of WCG's and ISF's.

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